

Landau's theorems for certain biharmonic mappings

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Published in Acta Mathematica Sinica, Chinese Series, 2011, Vol.54, No.1, 69-80.

Abstract: Let $f(z) = h(z) + \overline{g(z)}$ be a harmonic mapping of the unit disk U . In this paper, the sharp coefficient estimates for bounded planar harmonic mappings are established, the sharp coefficient estimates for normalized planar harmonic mappings with $|h(z)| + |g(z)| \leq M$ are also provided. As their applications, Landau's theorems for certain biharmonic mappings are provided, which improve and refine the related results of earlier authors.

Keywords: univalent; harmonic mapping; biharmonic mappings; Landau's theorem.

AMS Mathematics Subject Classification: Primary 30C99; Secondary 30C62.

1 Introduction and preliminaries

Suppose that $f(z) = u(z) + iv(z)$ is a two times continuously differentiable complex-valued function in a domain $D \subseteq \mathbb{C}$. Then $f(z)$ is harmonic function in a domain $D \subseteq \mathbb{C}$ if and only if $f(z)$ satisfies the following harmonic equation

$$\Delta f = 4f_{z\bar{z}} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0, \quad z = x + iy \in D,$$

where we use the common notations for its formal derivatives:

$$f_z = \frac{1}{2}(f_x - if_y), \quad f_{\bar{z}} = \frac{1}{2}(f_x + if_y).$$

A four times continuously differentiable complex-valued function $F(z) = U(z) + iV(z)$ is said to be a biharmonic in a domain $D \subseteq \mathbb{C}$ if and only if ΔF is harmonic in the domain D , i.e., $F(z)$ satisfies the following biharmonic equation

$$\Delta^2 F = \Delta(\Delta F) = 0, \quad z = x + iy \in D.$$

Biharmonic functions arise in many physical situations, particularly in fluid dynamics and elasticity problems, and have many important applications in engineering(see [1] for the details).

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This research is partly supported by the Natural Science Foundation of Fujian Province, China (No.2009J01007) and the Education Commission Foundation of Fujian Province, China (No.JA08013).

Notice that the composition $f \circ \phi$ of a harmonic function f with holomorphic function ϕ is harmonic, while this is not true when f is biharmonic. Without loss of generality, we consider the class of biharmonic mappings defined in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$.

For such function f , we define

$$\Lambda_f(z) = \max_{0 \leq \theta \leq 2\pi} |f_z(z) + e^{-2i\theta} f_{\bar{z}}(z)| = |f_z(z)| + |f_{\bar{z}}(z)|,$$

and

$$\lambda_f(z) = \min_{0 \leq \theta \leq 2\pi} |f_z(z) + e^{-2i\theta} f_{\bar{z}}(z)| = ||f_z(z)| - |f_{\bar{z}}(z)||.$$

We denote the Jacobian of f by J_f , Lewy[9] showed that a harmonic mapping $f(z)$ is locally univalent in a domain D if and only if $J_f(z) \neq 0$ for any $z \in D$. Of course, local univalence of f does not imply global univalence in a domain D . If D is simply connected, and $f(z)$ is a harmonic mapping in D , then $f(z)$ can be written as $f = f_1 + \overline{f_2}$ with $f(0) = f_1(0)$, f_1 and f_2 are analytic on D . Thus, we have

$$J_f(z) = |f_z(z)|^2 - |f_{\bar{z}}(z)|^2 = |f_1'(z)|^2 - |f_2'(z)|^2.$$

It is known [1] that a mapping F is biharmonic in a simply connected domain D if and only if F has the following representation:

$$F(z) = |z|^2 g(z) + h(z), \quad z \in D, \quad (1.1)$$

where $g(z)$ and $h(z)$ are complex-valued harmonic mappings in D . It is well known that $g(z)$ and $h(z)$ can be expressed as

$$g(z) = g_1(z) + \overline{g_2(z)}, \quad z \in D, \quad (1.2)$$

$$h(z) = h_1(z) + \overline{h_2(z)}, \quad z \in D, \quad (1.3)$$

where $g_1(z)$, $g_2(z)$, $h_1(z)$ and $h_2(z)$ are analytic in D (see [13] for the details).

The classical Landau theorem concerns determining the possibly largest schlicht disk for the properly normalized bounded analytic functions. It states that if f is an analytic function on the unit disk U with $f(0) = f'(0) - 1 = 0$ and $|f(z)| < M$ for $z \in U$, then f is univalent in the disk $|z| < r_0$ with $r_0 = 1/(M + \sqrt{M^2 - 1})$, and $f(|z| < r_0)$ contains a disk $|w| < R_0$ with $R_0 = Mr_0^2$. This result is sharp, with the extremal function $f(z) = Mz(1 - Mz)/(M - z)$ (see [3]).

For harmonic mappings in U , under suitable restriction, Chen et al. [4] obtained two versions of Landau's theorems. These versions are different from each other by the normalization conditions assumed for the bounded harmonic mapping. However, their results are not sharp. Better estimates were given in [6], late in [8, 11, 12].

For $r > 0$, we let U_r denote the disk with center at the origin and radius r . Abdulhadi and Muhanna [2] obtained two versions of Landau's theorems for biharmonic mappings; however, their results are not sharp.

Theorem A (Abdulhadi and Muhanna[2]) Let $f(z) = |z|^2 g(z) + h(z)$ be a biharmonic mapping of the unit disk \mathbb{U} , as in (1.1), with $f(0) = h(0) = J_f(0) - 1 = 0$ and $|g(z)| \leq M$, $|h(z)| \leq M$ for $z \in \mathbb{U}$. Then there is a constant $0 < \rho_1 < 1$ so that f is univalent in the disk \mathbb{U}_{ρ_1} . In specific ρ_1 satisfies the following equation

$$\frac{\pi}{4M} - 2\rho_1 M - \frac{2M\rho_1^2}{(1 - \rho_1)^2} - 2M \cdot \frac{2\rho_1 - \rho_1^2}{(1 - \rho_1)^2} = 0, \quad (1.4)$$

and $f(\mathbb{U}_{\rho_1})$ contains a schlicht disk \mathbb{U}_{R_1} with

$$R_1 = \frac{\pi}{4M}\rho_1 - 2M\frac{\rho_1^3 + \rho_1^2}{1 - \rho_1}. \quad (1.5)$$

Theorem B(Abdulahadi and Muhanna[2]) Let $g(z)$ be harmonic in the unit disk \mathbb{U} , with $g(0) = J_g(0) - 1 = 0$ and $|g(z)| \leq M$ for $z \in \mathbb{U}$. Then $f(z) = |z|^2g(z)$ is univalent in the disk \mathbb{U}_{ρ_2} with

$$\rho_2 = \frac{\pi}{\pi + 16M^2 + 2M\sqrt{2\pi + 64M^2}}, \quad (1.6)$$

and $f(\mathbb{U}_{\rho_2})$ contains a schlicht disk \mathbb{U}_{R_2} with

$$R_2 = \frac{\pi}{4M}\rho_2^3 - 2M\frac{\rho_2^4}{1 - \rho_2}. \quad (1.7)$$

Recently, Liu [10] established the following better coefficient estimates for bounded and normalized harmonic mappings.

Theorem C (Liu [10]) Suppose that $f(z) = h(z) + \overline{g(z)}$ is a harmonic mapping of the unit disk \mathbb{U} with $h(z) = \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$ for $z \in \mathbb{U}$.

(1) If $J_f(0) = 1$ and $|f(z)| < M$, then

$$|a_n| + |b_n| \leq \sqrt{2M^2 - 2}, \quad n = 2, 3, \dots, \quad (1.8)$$

and

$$\lambda_f(0) \geq \lambda_0(M) = \begin{cases} \frac{\sqrt{2}}{\sqrt{M^2-1} + \sqrt{M^2+1}}, & \text{if } 1 \leq M \leq M_0 = \frac{\pi}{2\sqrt[4]{2\pi^2-16}}, \\ \frac{\pi}{4M}, & \text{if } M > M_0 = \frac{\pi}{2\sqrt[4]{2\pi^2-16}} \approx 1.1296. \end{cases} \quad (1.9)$$

(2) If $\lambda_f(0) = 1$ and $|f(z)| < M$, then the inequalities (1.8) also hold.

By applying Theorem C, Liu improved Theorem A and Theorem B, and obtained some completely new results. Wang et al. [5] obtained two versions of Landau's theorems for biharmonic mappings of the form $L(F)$, where F is a biharmonic mapping, and $L = z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}$.

Theorem D (Liu [10]) Let $F(z) = |z|^2g(z) + h(z)$ be a biharmonic mapping of the unit disk U , as in (1.1), with $F(0) = h(0) = \lambda_F(0) - 1 = 0$ and $|g(z)| \leq M_1$, $|h(z)| \leq M_2$ for $z \in U$. Then, F is univalent in the disk U_{ρ_3} , and $F(U_{\rho_3})$ contains a schlicht disk U_{R_3} , where ρ_3 is the minimum positive root of the following equation

$$1 - 2rM_1 - 2M_1 \cdot \frac{r^2}{(1-r)^2} - \sqrt{2M_2^2 - 2} \cdot \frac{2r - r^2}{(1-r)^2} = 0,$$

and

$$R_3 = \rho_3 - \frac{2M_1\rho_3^3}{1 - \rho_3} - \sqrt{2M_2^2 - 2} \cdot \frac{\rho_3^2}{1 - \rho_3}.$$

Theorem E (Liu [10]) Let $F(z) = |z|^2g(z) + h(z)$ be a biharmonic mapping of the unit disk U , as in (1.1), with $F(0) = h(0) = J_F(0) - 1 = 0$, and $|g(z)| \leq M_1$, $|h(z)| \leq M_2$ for $z \in U$. Then F is univalent in the disk U_{ρ_4} , and $F(U_{\rho_4})$ contains a schlicht disk U_{R_4} , where ρ_4 is the minimum positive root of the following equation:

$$\lambda_0(M_2) - 2rM_1 - \frac{2M_1r^2}{(1-r)^2} - \sqrt{2M_2^2 - 2} \cdot \frac{2r - r^2}{(1-r)^2} = 0, \quad (1.10)$$

and

$$R_4 = \lambda_0(M_2)\rho_4 - \frac{2M_1\rho_4^3}{1-\rho_4} - \sqrt{2M_2^2-2} \cdot \frac{\rho_4^2}{1-\rho_4}, \quad (1.11)$$

where $\lambda_0(M)$ is defined by (1.9).

Theorem F (Liu [10]) Let $g(z)$ be harmonic in the unit disk U , with $g(0) = \lambda_g(0) - 1 = 0$ and $|g(z)| \leq M$ for $z \in U$. Then $F(z) = |z|^2g(z)$ is univalent in the disk U_{ρ_5} , and $F(U_{\rho_5})$ contains a schlicht disk U_{R_5} , where

$$\rho_5 = \frac{1}{1 + 2\sqrt{2M^2-2} + \sqrt{\sqrt{2M^2-2} + 8(M^2-1)}}, \quad (1.12)$$

and

$$R_5 = \begin{cases} \rho_5^3 - \sqrt{2M^2-2} \cdot \frac{\rho_5^4}{1-\rho_5}, & M > 1, \\ 1, & M = 1. \end{cases} \quad (1.13)$$

Above result is sharp when $M = 1$.

Theorem G (Liu [10]) Let $g(z)$ be harmonic in the unit disk \mathbb{U} , with $g(0) = J_g(0) - 1 = 0$ and $|g(z)| \leq M$ for $z \in \mathbb{U}$. Then $F(z) = |z|^2g(z)$ is univalent in the disk \mathbb{U}_{ρ_6} , and $F(\mathbb{U}_{\rho_6})$ contains a schlicht disk \mathbb{U}_{R_6} , where

$$\rho_6 = \frac{\lambda_0(M)}{\lambda_0(M) + 2\sqrt{2M^2-2} + \sqrt{\lambda_0(M)\sqrt{2M^2-2} + 8(M^2-1)}}, \quad (1.14)$$

and

$$R_6 = \begin{cases} \lambda_0(M)\rho_6^3 - \sqrt{2M^2-2} \cdot \frac{\rho_6^4}{1-\rho_6}, & \text{if } M > 1, \\ 1, & \text{if } M = 1, \end{cases} \quad (1.15)$$

where $\lambda_0(M)$ is defined by (1.9). Above result is sharp when $M = 1$.

The coefficient estimates (1.8) for bounded and normalized harmonic mappings in Theorem C are not sharp for $M > 1$. Furthermore, Liu[10] proposed the following conjecture:

Conjecture H (Liu [10]) Suppose that $f(z) = h(z) + \overline{g(z)}$ is a harmonic mapping of the unit disk U with $h(z) = \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$ are analytic on U , moreover $|f(z)| \leq M$ for $z \in U$. If $J_f(0) = 1$ or $\lambda_f(0) = 1$, then

$$|a_n| + |b_n| \leq M - \frac{1}{M}, \quad n = 2, 3, \dots,$$

with the extremal functions $f_n(z) = Mz((1 - Mz^{n-1})/(M - z^{n-1}))$ for $n = 2, 3, \dots$.

In this paper, we first establish the sharp coefficient estimates for bounded harmonic mappings (Lemma 2.1). Next, we prove that the above conjecture is true for the subclass of harmonic mappings $f(z) = h(z) + \overline{g(z)}$ which satisfy the following bounded condition

$$|h(z)| + |g(z)| \leq M. \quad (1.16)$$

Under this condition, we establish the sharp coefficient estimates for normalized harmonic mappings (Lemma 2.3, Corollary 2.4). Then, using these estimates, we verify several versions of Landau's theorem (Theorems 2.6, 2.6', 2.8, 2.8', 2.10, and 2.10', Corollary 2.12, 2.12'), which refine and improve Theorem A–G. In order to establish our main results, we recall the following lemma.

Lemma 1.1 (see, for example, p.35 in [7]) If $F(z) = a_0 + a_1z + \dots + a_nz^n + \dots$ is analytic and $|F(z)| \leq 1$ on U . Then $|a_n| \leq 1 - |a_0|^2$ for $n = 1, 2, \dots$.

2 Main results

We first establish the following sharp coefficient estimates for bounded harmonic mappings.

Lemma 2.1 Suppose that $f(z) = h(z) + \overline{g(z)}$ is a harmonic mapping of the unit disk U with $h(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$. If $|f(z)| \leq M$ for $z \in U$, then we have

$$|a_n| + |b_n| \leq \frac{4}{\pi} M, \quad n = 1, 2, \dots. \quad (2.1)$$

The inequalities (2.1) are sharp for all $n = 1, 2, \dots$.

Proof Fix $r \in (0, 1)$, we have

$$f(re^{i\theta}) = \sum_{n=0}^{\infty} a_n r^n e^{in\theta} + \sum_{n=1}^{\infty} \bar{b}_n r^n e^{-in\theta}, \quad \theta \in [0, 2\pi).$$

Hence we obtain

$$a_n r^n = \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) e^{-in\theta} d\theta, \quad n = 0, 1, 2, \dots, \quad (2.2)$$

and

$$\bar{b}_n r^n = \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) e^{in\theta} d\theta, \quad n = 1, 2, \dots. \quad (2.3)$$

For every n , we let $a_n = |a_n| e^{i\alpha_n}$, $b_n = |b_n| e^{i\beta_n}$, $\theta_n = \frac{\beta_n + \alpha_n}{2n}$. Since $|\cos n\theta|$ is a periodic function with period π , from (2.2), (2.3), we have

$$\begin{aligned} (|a_n| + |b_n|) r^n &= \left| \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) [e^{-i\alpha_n} e^{-in\theta} + e^{i\beta_n} e^{in\theta}] d\theta \right| \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})| |e^{-i\alpha_n} e^{-in\theta} + e^{i\beta_n} e^{in\theta}| d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})| |e^{-in\theta} + e^{i(\beta_n + \alpha_n)} e^{in\theta}| d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})| |e^{-in(\theta + \theta_n)} + e^{in(\theta + \theta_n)}| d\theta \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} 2M |\cos n(\theta + \theta_n)| d\theta \\ &= \frac{M}{\pi} \int_0^{2\pi} |\cos n\theta| d\theta = \frac{M}{n\pi} \int_0^{2n\pi} |\cos x| dx \\ &= \frac{4M}{\pi}. \end{aligned}$$

Setting $r \rightarrow 1^-$, we obtain $|a_n| + |b_n| \leq \frac{4M}{\pi}$ for all $n = 1, 2, \dots$.

In order to verify the sharpness of (2.1), we choose the harmonic mapping

$$F(z) = \operatorname{Im} \left\{ \frac{2M}{\pi} \log \frac{1+z}{1-z} \right\} = \frac{2M}{\pi} \arctan \frac{2y}{1-x^2-y^2}, \quad z = x + iy \in U.$$

Then we have $|F(z)| \leq M$ for $z \in U$. By a direct calculation, we obtain

$$F(z) = \sum_{n=0}^{\infty} A_n z^n + \overline{\sum_{n=1}^{\infty} B_n z^n} = -\frac{2Mi}{\pi} z + \dots + \overline{\frac{2Mi}{\pi} z} + \dots.$$

This implies $|A_1| + |B_1| = \frac{4M}{\pi}$. Let m be an integer number with $m > 1$. Then the mapping

$$F_m(z) = F(z^m) = \sum_{n=m}^{\infty} C_n z^n + \overline{\sum_{n=m}^{\infty} D_n z^n} = -\frac{2Mi}{\pi} z^m + \cdots + \overline{\frac{2Mi}{\pi} z^m} + \cdots$$

is harmonic in U with $|F_m(z)| \leq M$ for $z \in U$ such that $|C_m| + |D_m| = \frac{4M}{\pi}$. This completes the proof. \square

Remark 2.2 If $M > M'_0 = \frac{\pi}{\sqrt{\pi^2-8}} \approx 2.2976$, then $\sqrt{2M^2-2} > \frac{4M}{\pi}$. From Lemma 2.1, we improve Theorem C or Lemma 2.1 in [12] for $M > M'_0 = \frac{\pi}{\sqrt{\pi^2-8}} \approx 2.2976$.

Next, we establish the following sharp coefficient estimates for normalized harmonic mappings which satisfy the bounded condition (1.16).

Lemma 2.3 Suppose that $f(z) = h(z) + \overline{g(z)}$ is a harmonic mapping of the unit disk U with $h(z) = \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$ are analytic on U . If $|h(z)| + |g(z)| \leq M$ for $z \in U$, then we have $0 \leq \lambda_f(0) \leq M$, and

$$|a_n| + |b_n| \leq M - \frac{\lambda_f^2(0)}{M}, \quad n = 2, 3, \dots \quad (2.4)$$

Above estimates are sharp for all $n = 2, 3, \dots$, with the extremal functions $f_{a, n}(z)$ and $\overline{f_{a, n}(z)}$

$$f_{a, n}(z) = Mz \cdot \frac{a - Mz^{n-1}}{M - az^{n-1}}, \quad (2.5)$$

where $0 \leq a \leq M$.

Proof Fix $n \in \mathbb{N} - \{1\} = \{2, 3, \dots\}$, we choose a real number α such that $|a_n + e^{i\alpha}b_n| = |a_n| + |b_n|$, and set

$$l(z) = \frac{1}{M}[h(z) + e^{i\alpha}g(z)] = \sum_{k=1}^{\infty} \frac{a_k + e^{i\alpha}b_k}{M} z^k.$$

Since $h(z)$ and $g(z)$ are analytic and $|f(z)| \leq |h(z)| + |g(z)| \leq M$ on U , we get that $l(z)$ is analytic and $|l(z)| \leq (|h(z)| + |g(z)|)/M \leq 1$ on U . Notice $l(0) = 0$, by Schwarz lemma, we obtain that $|l(z)| \leq |z|$. Setting

$$F(z) = \frac{a_1 + e^{i\alpha}b_1}{M} + \sum_{k=2}^{\infty} \frac{a_k + e^{i\alpha}b_k}{M} z^{k-1},$$

it follows that $F(z)$ is analytic and $|F(z)| \leq 1$ on U . Since

$$||a_1| - |b_1|| = \lambda_f(0), \quad (2.6)$$

by Lemma 1.1 and (2.6), we obtain

$$\left| \frac{a_k + e^{i\alpha}b_k}{M} \right| \leq 1 - \left| \frac{a_1 + e^{i\alpha}b_1}{M} \right|^2 \leq 1 - \frac{||a_1| - |b_1||^2}{M^2} = 1 - \frac{\lambda_f^2(0)}{M^2}, \quad k = 2, 3, \dots$$

Thus we have $0 \leq \lambda_f(0) \leq M$. In particular, we have

$$|a_n| + |b_n| = |a_n + e^{i\alpha}b_n| \leq M \left(1 - \frac{\lambda_f^2(0)}{M^2} \right) = M - \frac{\lambda_f^2(0)}{M}.$$

Finally, it is obvious that the equalities hold for all $n = 2, 3, \dots$ for the functions

$$f_{a, n}(z) = Mz \frac{a - Mz^{n-1}}{M - az^{n-1}} = az - \left(M - \frac{a^2}{M} \right) z^n + \cdots$$

and

$$\overline{f_{a,n}(z)} = a\bar{z} - (M - \frac{a^2}{M})\bar{z}^n + \cdots,$$

respectively, where $a = \lambda_f(0)$. This completes the proof. \square

Setting $\lambda_f(0) = 1$ in Lemma 2.3, we get the following corollary.

Corollary 2.4 Suppose that $f(z) = h(z) + \overline{g(z)}$ is a harmonic mapping of the unit disk U with $h(z) = \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$ are analytic on U . If $|h(z)| + |g(z)| \leq M$ for $z \in U$, and $\lambda_f(0) = 1$, then $M \geq 1$, and

$$|a_n| + |b_n| \leq M - \frac{1}{M}, \quad n = 2, 3, \dots.$$

Above estimates are sharp for all $n = 2, 3, \dots$, with the extremal functions $f_{1,n}(z)$ and $\overline{f_{1,n}(z)}$ defined by (2.5).

Remark 2.5 Corollary 2.4 tell us that Conjecture H is true for the subclass of harmonic mappings f which satisfy the bounded condition (1.16) and $\lambda_f(0) = 1$. Setting $M = 1$ in Corollary 2.4, we get that $a_n = b_n = 0$ for $n = 2, 3, \dots$, thus $f(z) = \alpha z$ or $f(z) = \alpha \bar{z}$, where $|\alpha| = 1$.

Now, with the aid of Lemmas 2.1 and 2.3, we can improve Theorem D as follows.

Theorem 2.6 Let $F(z) = |z|^2 g(z) + h(z)$ be a biharmonic mapping of the unit disk U , as in (1.1)-(1.3), with $F(0) = h(0) = \lambda_F(0) - 1 = 0$, and $|g(z)| \leq M_1$, $|h(z)| \leq M_2$ for $z \in U$. Then F is univalent in the disk U_{r_1} , and $F(U_{\sigma_1})$ contains a schlicht disk U_{σ_1} , where r_1 is the minimum positive root of the following equation:

$$1 - 2rM_1 - \frac{4M_1 r^2}{\pi(1-r)^2} - K(M_2) \cdot \frac{2r - r^2}{(1-r)^2} = 0, \quad (2.7)$$

and

$$\sigma_1 = r_1 - \frac{4M_1 r_1^3}{\pi(1-r_1)} - K(M_2) \cdot \frac{r_1^2}{1-r_1}, \quad (2.8)$$

where $K(M_2) = \min\{\sqrt{2M_2^2 - 2}, \frac{4}{\pi}M_2\}$.

Proof From (1.2) and (1.3), we have

$$g(z) = g_1(z) + \overline{g_2(z)}, \quad h(z) = h_1(z) + \overline{h_2(z)},$$

where $g_1(z) = \sum_{n=0}^{\infty} a_n z^n$, $g_2(z) = \sum_{n=1}^{\infty} b_n z^n$, $h_1(z) = \sum_{n=1}^{\infty} c_n z^n$ and $h_2(z) = \sum_{n=1}^{\infty} d_n z^n$ are analytic in U , then, we have

$$\lambda_F(0) = |c_1| - |d_1| = \lambda_h(0) = 1. \quad (2.9)$$

To prove the univalence of $F(z)$ in U_{r_1} , we adopt the method used in [2, 10]. By means of the hypothesis of Theorem 2.5, by Theorem C and Lemmas 2.1, we have

$$|a_n| + |b_n| \leq \frac{4}{\pi}M_1 (n = 1, 2, \dots), \quad |c_n| + |d_n| \leq K(M_2) (n = 2, 3, \dots), \quad (2.10)$$

where $K(M_2) = \min\{\sqrt{2M_2^2 - 2}, \frac{4}{\pi}M_2\}$.

Thus, for $z_1 \neq z_2$ in U_r ($0 < r < r_1$), by (2.9) and (2.10), we have

$$\begin{aligned}
|F(z_1) - F(z_2)| &= \left| \int_{[z_1, z_2]} F_z(z) dz + F_{\bar{z}}(z) d\bar{z} \right| \\
&\geq \left| \int_{[z_1, z_2]} h_z(0) dz + h_{\bar{z}}(0) d\bar{z} \right| - \left| \int_{[z_1, z_2]} g(z)(\bar{z} dz + z d\bar{z}) \right| \\
&= \left| \int_{[z_1, z_2]} |z|^2 (g'_1(z) dz + \overline{g'_2(z)} d\bar{z}) \right| \\
&= \left| \int_{[z_1, z_2]} (h'_1(z) - h'_1(0)) dz + (\overline{h'_2(z)} - \overline{h'_2(0)}) d\bar{z} \right| \\
&\geq |z_1 - z_2| \left(\lambda_h(0) - 2rM_1 - \sum_{n=1}^{\infty} (|a_n| + |b_n|)nr^{n+1} - \sum_{n=2}^{\infty} (|c_n| + |d_n|)nr^{n-1} \right) \\
&\geq |z_1 - z_2| \left(1 - 2rM_1 - \sum_{n=1}^{\infty} \frac{4}{\pi} M_1 nr^{n+1} - \sum_{n=2}^{\infty} K(M_2) nr^{n-1} \right) \\
&\geq |z_1 - z_2| \left(1 - 2rM_1 - \frac{4M_1 r^2}{\pi(1-r)^2} - K(M_2) \cdot \frac{2r-r^2}{(1-r)^2} \right) > 0,
\end{aligned}$$

this implies $F(z_1) \neq F(z_2)$.

Notice that $F(0) = 0$, for any $z' = r_1 e^{i\theta} \in \partial U_{r_1}$, by (2.10), we have

$$\begin{aligned}
|F(z')| &\geq |c_1 z' + \bar{d}_1 \bar{z}'| - r_1^2 \left| \sum_{n=1}^{\infty} a_n z'^n + \bar{b}_n \bar{z}'^n \right| - \left| \sum_{n=2}^{\infty} c_n z'^n + \bar{d}_n \bar{z}'^n \right| \\
&\geq \lambda_h(0)r_1 - r_1^2 \sum_{n=1}^{\infty} (|a_n| + |b_n|)r_1^n - \sum_{n=2}^{\infty} (|c_n| + |d_n|)r_1^n \\
&\geq r_1 - r_1^2 \sum_{n=1}^{\infty} \frac{4}{\pi} M_1 r_1^n - \sum_{n=2}^{\infty} K(M_2) r_1^n \\
&= r_1 - \frac{4}{\pi} M_1 \frac{r_1^3}{1-r_1} - K(M_2) \frac{r_1^2}{1-r_1} = \sigma_1.
\end{aligned}$$

Hence, $F(z)$ is univalent on U_{r_1} and $F(U_{r_1})$ contains the disk U_{σ_1} , where r_1 is defined by (2.7) and σ_1 is defined by (2.8). This completes the proof of Theorem 2.6. \square

With the aid of Lemma 2.1 and Corollary 2.4, applying the similar method as in our proof of Theorem 2.6, we get a version of Landau's theorem as follows.

Theorem 2.6' Let $F(z) = |z|^2 g(z) + h(z)$ be a biharmonic mapping of the unit disk U , as in (1.1)-(1.3), with $F(0) = h(0) = \lambda_F(0) - 1 = 0$, and $|g(z)| \leq M_1$, $|h_1(z)| + |h_2(z)| \leq M_2$ for $z \in U$. Then F is univalent in the disk $U_{r'_1}$, and $F(U_{r'_1})$ contains a schlicht disk $U_{\sigma'_1}$, where r'_1 is the minimum positive root of the following equation:

$$1 - 2rM_1 - \frac{4M_1 r^2}{\pi(1-r)^2} - (M_2 - \frac{1}{M_2}) \cdot \frac{2r-r^2}{(1-r)^2} = 0,$$

and

$$\sigma_1 = r'_1 - \frac{4M_1 r'^3_1}{\pi(1-r'_1)} - (M_2 - \frac{1}{M_2}) \cdot \frac{r'^2_1}{1-r'_1}.$$

Remark 2.7 Notice that $\frac{4}{\pi}M_1 < 2M_1$, $M_2 - \frac{1}{M_2} < \frac{4}{\pi}M_2 \leq \sqrt{2M_2^2 - 2}$ for $M_2 \geq M'_0 = \frac{\pi}{\sqrt{\pi^2 - 8}} \approx 2.2976$, it is easy to verify that

$$r'_1 > r_1 > \rho_3, \quad \sigma'_1 > \sigma_1 > R_3.$$

With the aid of Theorem C and Lemma 2.1, applying the similar method as in our proof of Theorem 2.6, we can improve Theorem E as follows.

Theorem 2.8 Let $F(z) = |z|^2 g(z) + h(z)$ be a biharmonic mapping of the unit disk U , as in (1.1), with $F(0) = h(0) = J_F(0) - 1 = 0$, and $|g(z)| \leq M_1$, $|h(z)| \leq M_2$ for $z \in U$, as in (1.2) and (1.3). Then F is univalent in the disk U_{r_2} , and $F(U_{\sigma_2})$ contains a schlicht disk U_{σ_2} , where r_2 is the minimum positive root of the following equation:

$$\lambda_0(M_2) - 2rM_1 - \frac{4M_1 r^2}{\pi(1-r)^2} - K(M_2) \cdot \frac{2r - r^2}{(1-r)^2} = 0, \quad (2.11)$$

and

$$\sigma_2 = \lambda_0(M_2)r_2 - \frac{4M_1 r_2^3}{\pi(1-r_2)} - K(M_2) \cdot \frac{r_2^2}{1-r_2}, \quad (2.12)$$

where $\lambda_0(M)$ is defined by (1.9).

Proof By the proof of Theorem 2.6 and the hypothesis of Theorem 2.8, we have

$$J_F(0) = |c_1|^2 - |d_1|^2 = J_h(0) = 1.$$

By (1.9), we get that

$$\lambda_h(0) \geq \lambda_0(M_2). \quad (2.13)$$

By means of Lemmas 2.1 and Theorem C, applying the similar method as in our proof of Theorem 2.6, we can complete the proof of Theorem 2.8. \square

With the aid of Lemma 2.3 and (1.9), we can get the second version of Landau's theorem.

Theorem 2.8' Let $F(z) = |z|^2 g(z) + h(z)$ be a biharmonic mapping of the unit disk U , as in (1.1)-(1.3), with $F(0) = h(0) = J_F(0) - 1 = 0$, and $|g(z)| \leq M_1$, $|h_1(z)| + |h_2(z)| \leq M_2$ for $z \in U$. Then F is univalent in the disk $U_{r'_2}$, and $F(U_{\sigma'_2})$ contains a schlicht disk $U_{\sigma'_2}$, where r'_2 is the minimum positive root of the following equation:

$$\lambda_0(M_2) - 2rM_1 - \frac{4M_1 r^2}{\pi(1-r)^2} - (M_2 - \frac{\lambda_0^2(M_2)}{M_2}) \cdot \frac{2r - r^2}{(1-r)^2} = 0, \quad (2.14)$$

and

$$\sigma'_2 = \lambda_0(M_2)r'_2 - \frac{4M_1 r'^2_2}{\pi(1-r'_2)} - (M_2 - \frac{\lambda_0^2(M_2)}{M_2}) \cdot \frac{r'^2_2}{1-r'_2}, \quad (2.15)$$

where $\lambda_0(M)$ is defined by (1.9).

Remark 2.9 Notice that $\frac{4}{\pi}M_1 < 2M_1$, $M_2 - \frac{\lambda_0^2(M_2)}{M_2} < \frac{4}{\pi}M_2 \leq \sqrt{2M_2^2 - 2}$ for $M_2 \geq M'_0 = \frac{\pi}{\sqrt{\pi^2 - 8}} \approx 2.2976$, it is easy to verify that

$$r'_2 > r_2 > \rho_4, \quad \sigma'_2 > \sigma_2 > R_4,$$

for $M_2 > 1$.

The next theorem is different. Because, when $h = 0$, the Jacobian $J_F(0) = 0$ and hence we assume that $\lambda_g(0) = 1$ instead. With the aid of Theorem C and Lemma 2.1, we can improve Theorem F as follows.

Theorem 2.10 Let $g(z)$ be harmonic in the unit disk U , with $g(0) = \lambda_g(0) - 1 = 0$ and $|g(z)| \leq M$ for $z \in U$, as in (1.2). Then $F(z) = |z|^2 g(z)$ is univalent in the disk U_{r_3} , and $F(U_{r_3})$ contains a schlicht disk U_{σ_3} , with

$$K(M) = \min\{\sqrt{2M^2 - 2}, \frac{4M}{\pi}\}, \quad (2.16)$$

and

$$r_3 = \frac{1}{1 + 2K(M) + \sqrt{K(M) + 4K(M)^2}},$$

and

$$\sigma_3 = \begin{cases} r_3^3 - K(M) \cdot \frac{r_3^4}{1-r_3}, & M > 1, \\ 1, & M = 1. \end{cases}$$

Above result is sharp for $M = 1$.

Proof. If $F(z) = |z|^2 g(z)$ satisfies the hypothesis of Theorem 2.10, where

$$g(z) = g_1(z) + \overline{g_2(z)} = \sum_{n=1}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}$$

is harmonic in U , then

$$||a_1| - |b_1|| = \lambda_g(0) = 1. \quad (2.17)$$

By Theorem C and Lemma 2.1, we have

$$|a_n| + |b_n| \leq K(M) \quad (n = 2, 3, \dots). \quad (2.18)$$

For $z_1 \neq z_2$ in $U_r(0 < r < r_3)$, we have

$$F(z_1) - F(z_2) = \int_{[z_1, z_2]} F_z(z) dz + F_{\bar{z}}(z) d\bar{z} = \int_{[z_1, z_2]} (\bar{z}g + |z|^2 g'_1) dz + (zg + |z|^2 \overline{g'_2}) d\bar{z}$$

where $[z_1, z_2]$ is the line segment from z_1 to z_2 , $z = (1-t)z_1 + tz_2$ and $t \in [0, 1]$. Notice that r_3 is the minimum positive root of the following equation:

$$1 - K(M) \cdot \frac{4r - 3r^2}{(1-r)^2} = 0.$$

Applying the same method used in [2, 10], by (2.18) and (2.17), we have

$$\begin{aligned} \left| \frac{F(z_1) - F(z_2)}{z_1 - z_2} \right| &= \frac{1}{|z_1 - z_2|} \left| \int_{[z_1, z_2]} (\bar{z}g + |z|^2 g'_1) dz + (zg + |z|^2 \overline{g'_2}) d\bar{z} \right| \\ &\geq \int_{[z_1, z_2]} |z|^2 dt \cdot \left(||a_1| - |b_1|| - 2 \sum_{n=2}^{\infty} (|a_n| + |b_n|) r^{n-1} - \sum_{n=2}^{\infty} (|a_n| + |b_n|) nr^{n-1} \right) \\ &\geq \int_{[z_1, z_2]} |z|^2 dt \cdot \left(1 - 2 \sum_{n=2}^{\infty} K(M) r^{n-1} - \sum_{n=2}^{\infty} K(M) nr^{n-1} \right) \\ &= \int_{[z_1, z_2]} |z|^2 dt \cdot \left(1 - 2K(M) \cdot \frac{r}{1-r} - K(M) \cdot \frac{2r - r^2}{(1-r)^2} \right) \\ &= \int_{[z_1, z_2]} |z|^2 dt \cdot \left(1 - K(M) \cdot \frac{4r - 3r^2}{(1-r)^2} \right) > 0 \end{aligned}$$

this implies $F(z_1) \neq F(z_2)$ for $z_1 \neq z_2$ in $U_r(0 < r < r_3)$, where $|z| = |(1-t)z_1 + tz_2|$. Hence $F(z) = |z|^2 g(z)$ is univalent in the disk U_{r_3} .

From Theorem C, we get $M \geq 1$. When $M > 1$, by means of (2.17), for $|z| = r_3$, we have

$$\begin{aligned}
|F(z)| &= r_3^2 \left| \sum_{n=1}^{\infty} a_n z^n + \bar{b}_n \bar{z}^n \right| \\
&\geq r_3^2 |a_1 z + \bar{b}_1 \bar{z}| - r_3^2 \left| \sum_{n=2}^{\infty} a_n z^n + \bar{b}_n \bar{z}^n \right| \\
&\geq r_3^3 ||a_1| - |b_1|| - r_3^2 \sum_{n=2}^{\infty} (|a_n| + |b_n|) r_3^n \\
&\geq r_3^3 - r_3^2 \sum_{n=2}^{\infty} K(M) r_3^n \\
&= r_3^3 - K(M) \cdot \frac{r_3^4}{1 - r_3} = \sigma_3.
\end{aligned}$$

When $M = 1$, from Remark 2.5, we get that $g(z) = \alpha z$ or $g(z) = \alpha \bar{z}$ with $|\alpha| = 1$, thus for $|z| = r_3 = 1$,

$$|F(z)| = |z|^2 |\alpha| |z| = 1.$$

Hence, $F(U_{r_3})$ contains the disk U_{σ_3} , where r_3 is defined by (2) and σ_3 is defined by (2.17).

Finally, it is evident that $r_3 = \sigma_3 = 1$ for $M = 1$ is the best possible. This completes the proof of Theorem 2.10. \square

With the aid of Corollary 2.4, applying the similar method as in our proof of Theorem 2.10, we can get the following theorem.

Theorem 2.10' Let $g(z)$ be harmonic in the unit disk U , with $g(0) = \lambda_g(0) - 1 = 0$ and $|g_1(z)| + |g_2(z)| \leq M$ for $z \in U$, as in (1.2). Then $F(z) = |z|^2 g(z)$ is univalent in the disk $U_{r'_3}$, and $F(U_{r'_3})$ contains a schlicht disk $U_{\sigma'_3}$, where

$$r'_3 = \frac{1}{1 + 2(M - \frac{1}{M}) + \sqrt{M - \frac{1}{M} + 4(M - \frac{1}{M})^2}}, \quad (2.19)$$

and

$$\sigma'_3 = \begin{cases} r_3'^3 - (M - \frac{1}{M}) \cdot \frac{r_3'^4}{1 - r_3'}, & M > 1, \\ 1, & M = 1. \end{cases} \quad (2.20)$$

Above result is sharp for $M = 1$.

Remark 2.11 Notice that $M - \frac{1}{M} < K(M) \leq \sqrt{2M^2 - 2}$ for $M > 1$, and $K(M) = \frac{4M}{\pi} < \sqrt{2M^2 - 2}$ for $M > M'_0 = \frac{\pi}{\sqrt{\pi^2 - 8}} \approx 2.2976$, it is easy to verify that

$$r'_3 > r_3 \geq \rho_5, \quad \sigma'_3 > \sigma_3 \geq R_5,$$

for $M > 1$.

With the aid of Theorem C and Lemma 2.1, applying the similar method as in our proof of Theorem 2.10, we can improve Theorem G as follows.

Corollary 2.12 Let $g(z)$ be harmonic in the unit disk U , with $g(0) = J_g(0) - 1 = 0$ and $|g(z)| \leq M$ for $z \in U$, as in (1.2). Then $F(z) = |z|^2 g(z)$ is univalent in the disk U_{r_4} , and $F(U_{r_4})$ contains a schlicht disk U_{σ_4} , where

$$r_4 = \frac{\lambda_0(M)}{\lambda_0(M) + 2K(M) + \sqrt{\lambda_0(M)K(M) + 4K(M)^2}},$$

and

$$\sigma_4 = \begin{cases} \lambda_0(M) r_4^3 - K(M) \cdot \frac{r_4^4}{1-r_4}, & \text{if } M > 1, \\ 1, & \text{if } M = 1, \end{cases}$$

where $\lambda_0(M)$ is defined by (1.9), and $K(M)$ is defined by (2.16). Above result is sharp for $M = 1$.

With the aid of Lemma 2.3, if we apply the same method as in our proof of Theorem 2.10, we obtain the fourth version of Landau's theorem for biharmonic mappings as follows.

Corollary 2.12' Let $g(z)$ be harmonic in the unit disk U , with $g(0) = J_g(0) - 1 = 0$ and $|g_1(z)| + |g_2(z)| \leq M$ for $z \in U$, as in (1.2). Then $F(z) = |z|^2 g(z)$ is univalent in the disk $U_{r'_4}$, and $F(U_{r'_4})$ contains a schlicht disk $U_{\sigma'_4}$, where

$$r'_4 = \frac{\lambda_0(M)}{\lambda_0(M) + 2(M - \frac{\lambda_0^2(M)}{M}) + \sqrt{\lambda_0(M)(M - \frac{\lambda_0^2(M)}{M}) + 4(M - \frac{\lambda_0^2(M)}{M})^2}},$$

and

$$\sigma'_4 = \begin{cases} \lambda_0(M) r_4'^3 - (M - \frac{\lambda_0^2(M)}{M}) \cdot \frac{r_4'^4}{1-r_4'}, & \text{if } M > 1, \\ 1, & \text{if } M = 1, \end{cases}$$

where $\lambda_0(M)$ is defined by (1.9). Above result is sharp for $M = 1$.

Remark 2.13 Notice that $M - \frac{\lambda_0^2(M)}{M} < K(M) \leq \sqrt{2M^2 - 2}$ for $M > 1$, and $\lambda_1(M) = \frac{4M}{\pi} < \sqrt{2M^2 - 2}$ for $M > M'_0 = \frac{\pi}{\sqrt{\pi^2 - 8}} \approx 2.2976$, it is easy to verify that

$$r'_4 > r_4 > \rho_6 > \rho_2, \quad \sigma'_4 > \sigma_4 > R_6 > R_2,$$

for $M > M'_0 = \frac{\pi}{\sqrt{\pi^2 - 8}} \approx 2.2976$.

References

- [1] Z. Abdulhadi, Y. Muhanna, S. Khuri, On univalent solutions of the biharmonic equations, *J. Inequal. Appl.* 2005(5) (2005), pp.469-478.
- [2] Z. Abdulhadi, Y. Muhanna, Landau's theorems for biharmonic mappings, *J. Math. Anal. Appl.* 338(2008), pp.705-709.
- [3] H.-H. Chen, On the Bloch constant. In: Approximation Complex Analysis, and Potential Theory. Dordrecht: KLuwer Acad Publ, 2001, 129-161.
- [4] H.-H. Chen, P. M. Gauthier and W. Hengartner, Bloch constants for planar harmonic mappings, *Proc. Amer. Math. Soc.* 128(2000), pp.3231-3240.
- [5] Sh. Chen, S. Ponnusamy and X. Wang, Landau's theorems for certain biharmonic mappings, *Applied Math. Computation.* 208(2009), pp.427-433.
- [6] M. Dorff, M. Nowak, Landau's theorem for planar harmonic mappings, *Comput. Meth. Funct. Theory* 4(2000), pp.151-158.
- [7] I. Graham, G.Kohr, Geometric Function Theory in One and Higer Dimensions, Dekker, New York,2003.
- [8] A. Grigoryan, Landau and Bloch theorems for harmonic mappings, *Complex Variable Theory Appl.* 51(1) (2006), pp.81-87.

- [9] H. Lewy, On the non-vanishing of the Jacobian in certain one-to-one mappings, *Bull. Amer. Math. Soc.* 42(1936), pp.689-692.
- [10] M.-S. Liu, Landau's theorems for biharmonic mappings, *Complex Variables and Elliptic Equations*. 53(9) (2008), pp.843-855.
- [11] M.-S. Liu, Estimates on Bloch constants for planar harmonic mappings, *Sci. China Ser. A-Math.* 52(1) (2009), pp.87-93.
- [12] M.-S. Liu, Landau's theorem for planar harmonic mappings, *Computers and Mathematics with Applications*. 57(7) (2009), pp.1142-1146.
- [13] Y. Muhanna, G. Schober, Harmonic mappings onto convex mapping domains, *Canad. J. Math.* XXXIX (6) (1987), pp.1489-1530.